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The Landauer Resistivity on Quantum Wires

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We study the Landauer resistivity of the Kronig–Penney model which has various behavior depending on the potential and the Fermi energy. In the case of the Sturmian quasiperiodic potential, we discuss examples in which lim inf of it is zero.

KEY WORDS: Kronig–Penney model; Landauer resistivity; quasiperiodic potential.

1. INTRODUCTION

The subject of this paper is the Landauer resistivity of the Kronig–Penney model defined by

$$H := -\frac{d^2}{dx^2} + \sum_{j \in \mathbb{Z}} V(j)\, \delta(x-j), \qquad \text{on } L^2(\mathbb{R}).$$

More precisely, H is the Laplacian with boundary conditions on integer points:

$$H := -\frac{d^2}{dx^2} \qquad \text{on } \mathscr{D},$$

where

$$\mathscr{D} := \{\psi \in H^1(\mathbb{R}) \cap H^2(\mathbb{R} \setminus \mathbb{Z}) : \psi'(j+) - \psi'(j-) = V(j)\, \psi(j),\, j \in \mathbb{Z}\}.$$

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$H^p(\Omega)$ is the Sobolev space of order p on Ω and $V(j) \in \mathbb{R}$, $j \in \mathbb{Z}$. H is self-adjoint[1, 2] and can be regarded as a model describing non-interacting electrons on the quantum wire. We would like to consider the Landauer resistivity of H which, however, is defined on Hamiltonians with compactly supported potentials. Thus we first consider the truncated Hamiltonian so that it has n δ-barriers.

$$H_n := -\frac{d^2}{dx^2} + \sum_{j=1}^{n} V(j)\,\delta(x-j), \qquad \text{on } L^2(\mathbb{R}).$$

In other words, H_n is the Laplacian on the domain

$$\mathscr{D}_n := \{\psi \in H^1(\mathbb{R}) \cap H^2(\mathbb{R}\setminus Y_n) : \psi'(j+) - \psi'(j-) = V(j)\,\psi(j),\ j \in Y_n\},$$

where $Y_n := \{1, 2,..., n\}$. We fix the Fermi energy $\epsilon_F > 0$ arbitrary and set $k = \sqrt{\epsilon_F}$. The Jost solution ψ of the equation $H_n \psi = \epsilon_F \psi$ is defined so that it satisfies following condition.

$$\psi(x) = \begin{cases} ce^{ikx} + de^{-ikx}, & (\text{if } x < 1), \qquad c, d \in \mathbb{C}, \\ e^{ikx}, & (\text{if } x > n). \end{cases} \tag{1.1}$$

We do not consider the case where $\epsilon_F \leqslant 0$, for the Jost solution can not be defined. $c = c(n, \epsilon_F)$, $d = d(n, \epsilon_F)$ are determined by condition (1.1) and the transmission probability $\tau(n, \epsilon_F)$ and the Landauer resistivity $\rho_L(n, \epsilon_F)$ are defined by[3]

$$\tau(n, \epsilon_F) := \frac{1}{|c(n, \epsilon_F)|^2}, \qquad \rho_L(n, \epsilon_F) := \frac{1 - \tau(n, \epsilon_F)}{\tau(n, \epsilon_F)}.$$

We derive an explicit representation of $\rho_L(n, \epsilon_F)$ in terms of the transfer matrix of H in the next section. It turns out that $1 \leqslant |c(n, \epsilon_F)| < \infty$ and hence $\tau(n, \epsilon_F)$ and $\rho_L(n, \epsilon_F)$ are always well-defined. If $\lim_{n \to \infty} \rho_L(n, \epsilon_F)$ exists, it may be reasonable to regard it as the electrical resistivity of H corresponding to the Fermi energy $\epsilon_F > 0$. This motivates us to study the behavior of $\rho_L(n, \epsilon_F)$ as n tends to infinity, which is the purpose of this paper. Some of them are review of known facts, while that in the case of quasiperiodic potential is new.

In Section 2, we introduce the transfer matrix and compute the transmission probability $\tau(n, \epsilon_F)$ and the Landauer resistivity $\rho_L(n, \epsilon_F)$ explicitly.

In Sections 3–5, we review spectral properties of H and study the behavior of $\rho_L(n, \epsilon_F)$ as n tends to infinity, when V is periodic, random, and Sturmian quasiperiodic respectively. Section 6 is a summary of results.

2. PRELIMINARIES

First of all, we introduce the transfer matrix of H. The solution ψ to the equation $H\psi = k^2\psi$ ($k > 0$, $k^2 = \epsilon_F$) has the following form.

$$\psi(x) = C_j e^{ik(x-j)} + D_j e^{-ik(x-j)}, \qquad x \in (j, j+1), \quad C_j, D_j \in \mathbb{C}. \tag{2.1}$$

By the boundary condition, ${}^t(C_j, D_j)$ satisfies

$$\begin{pmatrix} C_j \\ D_j \end{pmatrix} = T(j, k^2) \begin{pmatrix} C_{j-1} \\ D_{j-1} \end{pmatrix}, \qquad j \in \mathbb{Z},$$

where the (elementary) transfer matrix $T(j, k^2)$ is given by

$$T(j, k^2) := \begin{pmatrix} \left(1 + \frac{V(j)}{2ik}\right) e^{ik} & \frac{V(j)}{2ik} e^{-ik} \\ -\frac{V(j)}{2ik} e^{ik} & \left(1 - \frac{V(j)}{2ik}\right) e^{-ik} \end{pmatrix}.$$

Let $\mathcal{G}$ be a subgroup of $SL(2, \mathbb{C})$

$$\mathcal{G} := \left\{ \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} : \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1 \right\}. \tag{2.2}$$

Then $T(j, k^2) \in \mathcal{G}$ and we can write

$$\begin{aligned} W(n, k^2) &:= T(n, k^2)\, T(n-1, k^2) \cdots T(1, k^2) \\ &= \begin{pmatrix} a(n, k^2) & b(n, k^2) \\ \overline{b(n, k^2)} & \overline{a(n, k^2)} \end{pmatrix}, \end{aligned}$$

where $a(n, k^2), b(n, k^2) \in \mathbb{C}$, $|a(n, k^2)|^2 - |b(n, k^2)|^2 = 1$. According to (1.1), we need to derive $c = c(n, k^2)$, $d = d(n, k^2) \in \mathbb{C}$ such that

$$\begin{pmatrix} e^{ikn} \\ 0 \end{pmatrix} = W(n, k^2) \begin{pmatrix} c \\ d \end{pmatrix}.$$

Direct computation gives

$$\tau(n, k^2) = \frac{1}{|a(n, k^2)|^2}, \qquad \rho_L(n, k^2) = |b(n, k^2)|^2. \tag{2.3}$$

Hence the Landauer resistivity $\rho_L(n, k^2)$ is equal to square of the absolute value of $(1, 2)$-element of $W(n, k^2)$. We note $\rho_L(n, k^2)$ is related to the Hilbert–Schmidt norm of $W(n, k^2)$.

$$\rho_L(n, k^2) = \frac{\|W(n, k^2)\|_{\mathrm{HS}}^2 - 2}{4}, \tag{2.4}$$

where $\|A\|_{\mathrm{HS}} = \sqrt{\mathrm{tr}(A^*A)}$ is the Hilbert–Schmidt norm. $\rho_L(n, k^2)$ is also related to the generalized eigenfunctions of H. In fact, $b(n, k^2) = \frac{1}{2}(\psi(j) + \frac{\psi'(j+)}{ik})$ where ψ is the solution to $H\psi = k^2\psi$ with the initial condition $\psi(0) = -\frac{\psi'(0+)}{ik} = 1$.

3. PERIODIC POTENTIAL

In the rest of this paper, we compute $\rho_L(n, \epsilon_F)$ and study its behavior as n tends to infinity. In this section, we consider the case in which V $(\not\equiv 0)$ is periodic with period L: $V(j) = V(j+L)$, $j \in \mathbb{Z}$. By the direct integral decomposition and analytic perturbation theory, the spectrum of H is seen to be absolutely continuous[2] which has the following representation.

$$\sigma(H) = \{E \in \mathbb{R} : |D(E)| \leqslant 2\},$$

where $D(E) := \mathrm{tr}\, W(L, E)$ is the discriminant. In Section 2, we defined $T(j, E)$ only when $E > 0$. When $E < 0$, the branch of $k = \sqrt{E}$ is taken so that $\Im\sqrt{E} \geqslant 0$. When $E = 0$, we take different basis of the solution to $H\psi = E\psi$. The behavior of $\rho_L(n, k^2)$ as $n \to \infty$ is summarized as follows.

Theorem 3.1 (periodic case).

(1) If $\epsilon_F \in \rho(H)$, then $\rho_L(n, \epsilon_F)$ diverges exponentially.

(2) If $|D(\epsilon_F)| \leqslant 2$, then for except at most countable set $A(\subset \mathbb{R})$, we have

(i) if $|D(\epsilon_F)| = 2$, then $\rho_L(n, \epsilon_F)$ diverges like $O(n^2)$ as $n \to \infty$.

(ii) if $|D(\epsilon_F)| < 2$, then $\rho_L(n, \epsilon_F) = O(1)$ as $n \to \infty$ and does not converge. Moreover, for a.e. ϵ_F in $\sigma(H)$, $\overline{\{\rho_L(n, \epsilon_F); n \in \mathbb{N}\}} = \bigcup_{j=1}^{L} [c_j(\epsilon_F),$

$d_j(\epsilon_F)]$ for some $0 \leqslant c_j(\epsilon_F) < d_j(\epsilon_F)$, $j = 1, \ldots, L$. Otherwise $\{\rho_L(n, \epsilon_F); n \in \mathbb{N}\}$ consists of finitely many points.

(3) if $\epsilon_F \in A$, then $\{\rho_L(n, \epsilon_F); n \in \mathbb{N}\}$ consists of finitely many points.

Remark 3.1. The statement of Theorem 3.1 becomes clear in many specific cases. When $V(j) = V$ $(\neq 0)$ is constant for instance, $A = \emptyset$. $|D(\epsilon_F)| = 2$ if and only if $\epsilon_F \in \partial\sigma(H)$, and $|D(\epsilon_F)| < 2$ if and only if $\epsilon_F \in \sigma(H)^\circ$ ($\sigma(H)^\circ$ is the set of interior points of $\sigma(H)$). In the case of (2)(ii), $c(\epsilon_F) = 0$ and $d(\epsilon_F) = V^2/\epsilon_F(4 - D(\epsilon_F))$.

Remark 3.2. The dichotomy in (2)(ii) in Theorem 3.1 originates in that $\theta/\pi \in \mathbb{Q}$ or $\in \mathbb{Q}^c$, where $2\cos\theta = D(\epsilon_F)$, $\theta \in (0, \pi)$. In these cases, the corresponding generalized eigenfunction is periodic or quasiperiodic respectively.

Proof. We first consider the points where $n = jL$, $j \in \mathbb{N}$. Then the problem is reduced to the computation of $(1, 2)$-element of $W(L, E)^j$ $(n = jL)$. Let λ, λ^{-1} $(\in \mathbb{C})$ be eigenvalues of the matrix $W(L, E)$. We have the following expressions for $W(L, E)^j$. If $|\mathrm{tr}\, W(L, E)| \neq 2$, then $\lambda \neq \lambda^{-1}$ and

$$W(L, E)^j = \frac{\lambda^j - \lambda^{-j}}{\lambda - \lambda^{-1}} W(L, E) - \frac{\lambda^{j-1} - \lambda^{-j+1}}{\lambda - \lambda^{-1}} I, \tag{3.1}$$

where I is the 2×2-identity matrix. If $\mathrm{tr}\, W(L, E) = \pm 2$, then $W(L, E)$ has a multiple eigenvalue $\lambda = \pm 1$ respectively and

$$W(L, E)^j = j\lambda^{j-1} W(L, E) - (j-1)\, \lambda^j I. \tag{3.2}$$

By (3.1), (3.2), we compute $\rho_L(jL, \epsilon_F)$.

Case (a). $|D(\epsilon_F)| > 2$: Let $\lambda \in \mathbb{R}$ be the eigenvalue of $W(L, E)$ with $|\lambda| > 1$. Then $\lambda = (\mathrm{sgn}\, \lambda)\, e^{\theta}$ for some $\theta > 0$ and by (2.3),

$$\rho_L(jL, \epsilon_F) = \left(\frac{\sinh j\theta}{\sinh \theta}\right)^2 |b(L, \epsilon_F)|^2, \qquad j \in \mathbb{Z}. \tag{3.3}$$

We note, since $W(L, E) \in \mathscr{G}$ and $|\mathrm{tr}\, W(L, E)| > 2$, $b(L, \epsilon_F) \neq 0$.

Case (b). $|D(\epsilon_F)| = 2$: $\lambda = \pm 1$ and

$$\rho_L(jL, \epsilon_F) = j^2\, |b(L, \epsilon_F)|^2. \tag{3.4}$$

Case (c). $|D(\epsilon_F)|<2$: $\lambda=e^{i\theta}$ for some $\theta\in(0,\pi)$ and

$$\rho_L(jL,\epsilon_F)=\left(\frac{\sin j\theta}{\sin\theta}\right)^2|b(L,\epsilon_F)|^2,\qquad j\in\mathbb{Z}. \tag{3.5}$$

When $|D(\epsilon_F)|>2$, (3.3) implies $b(jL,\epsilon_F)$ diverges exponentially as $j\to\infty$. For intermediate points, that is, points of the form: $n=jL+l$, $(1\leqslant l\leqslant L-1)$, we can see $\|W(n,\epsilon_F)\|_{\mathrm{op}}$ diverges exponentially ($\|\cdot\|_{\mathrm{op}}$ is the operator norm), or alternatively we can use positivity of the Lyapunov exponent discussed in later sections.

When $|D(\epsilon_F)|\leqslant 2$, it is possible that $b(L,\epsilon_F)=0$ (a simple example is: $L=2$, $V(1)+V(2)=0$, and $\epsilon_F=(n\pi)^2$, $n\in\mathbb{N}$). This leads us to set

$$A:=\{\epsilon_F\in\sigma(H):b(L,\epsilon_F)=0\},$$

which consists of at most countably many isolated points. Since the statement of theorem is obvious for $\epsilon_F\in A$, let $\epsilon_F\notin A$. When $|D(\epsilon_F)|=2$, (3.4) implies $\rho_L(jL,\epsilon_F)=j^2\,|b(L,\epsilon_F)|^2$ which diverges in the order of n^2. Moreover, Lemma 3.1 given below shows $b(jL+l,\epsilon_F)=\alpha_l\,j+\beta_l$ for some α_l, $\beta_l\in\mathbb{C}$, and $\alpha_l\neq 0$ unless $b(L,\epsilon_F)=0$. When $|D(\epsilon_F)|<2$, direct computation gives $b(jL+l,\epsilon_F)=\gamma_l\sin j\theta+\delta_l\cos j\theta$ for some γ_l, $\delta_l\in\mathbb{C}$, $\gamma_l\delta_l\neq 0$.

Now the statement of Theorem 3.1 follows immediately from these considerations. ▌

Lemma 3.1. Suppose $\epsilon_F\notin A$ and $|D(\epsilon_F)|=2$. Then $b(jL+l)=\alpha_l\,j+\beta_l$ for some α_l, $\beta_l\in\mathbb{C}$, $\alpha_l\neq 0$.

Proof. We suppose $\lambda=1$. The proof for $\lambda=-1$ is similar. Letting $B(l,\epsilon_F):=T(l,\epsilon_F)\,T(l-1,\epsilon_F)\cdots T(1,\epsilon_F)$ $(1\leqslant l\leqslant L-1)$, we have

$$b(jL+l,\epsilon_F)=j(BW(1,2)-b(l,\epsilon_F))+b(l,\epsilon_F),$$

where $BW(1,2)$ is the $(1,2)$-element of the matrix $B(l,\epsilon_F)\,W(L,\epsilon_F)$. Hence $b(jL+l,\epsilon_F)=\alpha_l\,j+\beta_l$ for some α_l, $\beta_l\in\mathbb{C}$. Next we suppose $BW(1,2)=b(l,\epsilon_F)$ and would like to deduce $b(L,\epsilon_F)=0$. Since $W(L,\epsilon_F)$, $B(l,\epsilon_F)\in\mathscr{G}$, we can write

$$W(L,\epsilon_F)=\begin{pmatrix}\alpha_w, & \beta_w,\\ \overline{\beta_w} & \overline{\alpha_w}\end{pmatrix},\qquad \alpha_w,\beta_w\in\mathbb{C},\quad |\alpha_w|^2-|\beta_w|^2=1,$$

$$B(l,\epsilon_F)=\begin{pmatrix}\gamma_l & \delta_l\\ \overline{\delta_l} & \overline{\gamma_l}\end{pmatrix},\qquad \gamma_l,\delta_l\in\mathbb{C},\quad |\gamma_l|^2-|\delta|^2=1.$$

Since $\lambda = 1$ and $\epsilon_F \notin A$, $\alpha_w = 1 + ia$ for some $a \in \mathbb{R}$, $a \neq 0$. $BW(1,2) = b(l, \epsilon_F)$ means $\gamma_l \beta_w + \delta_l \overline{\alpha_w} = \delta_l$ and thus

$$\delta_l = \frac{\gamma_l \beta_w}{1 - \overline{\alpha_w}} = \frac{\gamma_l \beta_w}{ia}.$$

Substituting it to the equation $|\gamma_l|^2 - |\delta_l|^2 = 1$, we must have

$$|\gamma_l|^2 \left(1 - \frac{|\beta_w|^2}{a^2}\right) = 1.$$

However, $|\beta_w|^2 = |\alpha_w|^2 - 1 = a^2$ implies LHS $= 0$. ∎

Remark 3.3. If $\epsilon_F \in \mathscr{E} := \{(n\pi)^2 : n \in \mathbb{N}\}$ in which case $|D(\epsilon_F)| = 2$, the computation becomes easier. In fact, $\{T(j, \epsilon_F)\}_{j \in \mathbb{Z}}$ commutes each other so that we have $b(n, \epsilon_F) = \frac{1}{2i\sqrt{\epsilon_F}} \sum_{j=1}^{n} V(j)$. Hence $\lim_{n \to \infty} \frac{\rho_L(n, \epsilon_F)}{n^2} = \frac{1}{4\epsilon_F} (\frac{1}{L} \sum_{j=1}^{L} V(j))^2$. We note, according to the relation found in ref. 4, $\sigma(H) \cap \mathscr{E}$ is related to that of the free Laplacian on $l^2(\mathbb{Z})$,[5] while the computation above implies the behavior of wave functions are not.

4. RANDOM POTENTIAL

Let $\{V(j)\}_{j \in \mathbb{Z}} = \{V_\omega(j)\}_{j \in \mathbb{Z}}$ be the independent, identically distributed random variables on a probability space $(\Omega, \mathscr{F}, \mathbf{P})$. We assume the distribution of $V_\omega(0)$ has the density function which is bounded and compactly supported contained in the positive real line. That is, there exists constants c_1, c_2 such that $0 < c_1 \leqslant V_\omega(0) \leqslant c_2 < \infty$, a.s. Then[6] $\sigma(H) = \Sigma$, a.s. where

$$\Sigma = \left\{ E \in (0, \infty) : \left| 2 \cos \sqrt{E} + V_{\inf} \frac{\sin \sqrt{E}}{\sqrt{E}} \right| \leqslant 2 \right\}, \qquad V_{\inf} := \text{ess-} \inf_{\omega \in \Omega} V_\omega(0).$$

Moreover, the spectrum of H on $\mathscr{E}^c$ is almost surely pure point with exponentially decaying eigenfunctions (Anderson localization). The positivity of the Lyapunov exponent for $\epsilon_F \notin \mathscr{E}$ guaranteed by Furstenberg's theorem[7] and Remark 3.3 for $\epsilon_F \in \mathscr{E}$ give the following result.

Theorem 4.1 (random case).

(1) If $\epsilon_F \notin \mathscr{E}$, then $\rho_L(n, \epsilon_F)$ diverges exponentially **P**-a.s.

(2) If $\epsilon_F \in \mathscr{E}$, then $\lim_{n \to \infty} \frac{\rho_L(n, \epsilon_F)}{n^2} = \frac{1}{4\epsilon_F} (\mathbf{E} V_\omega(0))^2$, **P**-a.s. where **E** stands for taking expectations.

Remark 4.1. In refs. 8–11, they considered the charge transport on the multidimensional Anderson model and showed it is zero almost surely which is consistent with the theorem above.

5. QUASIPERIODIC POTENTIAL

In this section, we consider the following quasiperiodic potential

$$V(j) = V_\theta(j) := \lambda_1 \chi_A(\Phi(\alpha j)+\theta) + \lambda_2 \chi_{A^c}(\Phi(\alpha j)+\theta), \qquad j \in \mathbb{Z},$$

where $\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1 \neq \lambda_2$, $A = [1-\alpha, 1) \subset \mathbb{R}/\mathbb{Z}$, $\alpha \in (0, 1) \cap \mathbb{Q}^c$, and $\theta \in \mathbb{R}/\mathbb{Z}$. Φ: $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$ is the canonical projection. The spectrum of H is purely singular continuous and is a Cantor set (i.e., nowhere dense closed set without isolated points) for (λ_1, λ_2)-a.e. in $\mathbb{R}^2$ and for any $\theta \in \mathbb{R}/\mathbb{Z}$.[12–16] Moreover, the spectral measure is absolutely continuous w.r.t. $\mathscr{H}^\beta$: the Hausdorff measure of dimension β for some $\beta > 0$ (which follows from arguments in refs. 17–19), if α is a bounded density number. To state the results below, we consider the continued fraction expansion of α.

$$\alpha = [0, a_1(\alpha), a_2(\alpha), \ldots] := \cfrac{1}{a_1(\alpha) + \cfrac{1}{a_2(\alpha) + \ddots}}, \qquad a_n(\alpha) \in \mathbb{N}.$$

The associated rational approximation p_n/q_n satisfies[20]

$$p_{n+1} = a_{n+1}(\alpha)\, p_n + p_{n-1}, \tag{5.1}$$

$$q_{n+1} = a_{n+1}(\alpha)\, q_n + q_{n-1}, \qquad n \geqslant 0, \tag{5.2}$$

with $p_0 = 0$, $q_0 = 1$, $p_{-1} = 1$, $q_{-1} = 0$. We say α is a bounded density number if $\limsup_{n\to\infty} \frac{1}{n} \sum_{j=1}^n a_j(\alpha) < \infty$ (a typical example is the golden number: $\alpha = (-1+\sqrt{5})/2$). A simple guess by the fact $\sigma(H) = \partial\sigma(H)$ and argument in Theorem 3.1 leads us to a speculation that $\rho_L(n, k^2)$ would grow polynomially. However, the situation may be more complicated as will be discussed later.

Theorem 5.1 (quasiperiodic case).

(1) If $\epsilon_F \in \rho(H)$, then $\rho_L(n, \epsilon_F)$ diverges exponentially.

(2) (refs. 21–24) If $\epsilon_F \in \sigma(H)$ and if α is a bounded density number, then $\rho_L(n, \epsilon_F)$ grows at most polynomial order.

(3) If $\epsilon_F \in \mathscr{E}$, then $\lim_{n\to\infty} \frac{\rho_L(n, \epsilon_F)}{n^2} = \frac{1}{4\epsilon_F}(\alpha\lambda_1 + (1-\alpha)\, \lambda_2)^2$.

Remark 5.1.

(1) This result contrasts with those in previous sections. The behavior of $\rho(n)$ is known to be complicated.[21]

(2) Spectral properties of H is related to the behavior of tr $W(q_n, E)$ as n tends to ∞. In fact, $\sigma(H)$ coincides with the set where the sequence $\{\mathrm{tr}\, W(q_n, E)\}_{n=1}^{\infty}$ $(=\{2\Re a(q_n, E)\}_{n=1}^{\infty})$ is bounded.[14, 5] On the other hand, the behavior of $\{\rho_L(q_n, \epsilon_F)\}_{n=1}^{\infty}$ is related to that of $\{\Im a(q_n, \epsilon_F)\}_{n=1}^{\infty}$. Therefore we may say that the Landauer resistivity reflects some aspects of the systems which is different from spectral properties.

(3) If V is of almost Mathieu type (that is, $V_\theta(j) = \lambda \cos(2\pi(\alpha j + \theta))$, $\lambda \in \mathbb{R}$, $\alpha \notin \mathbb{Q}$, $\theta \in \mathbb{R}/\mathbb{Z}$), by Herman's theorem,[25, 26] the Lyapunov exponent is positive if $\epsilon_F \in B := \{E \in \mathbb{R} : |\lambda \sin \sqrt{E}/\sqrt{E}| > 2\}$. Hence $\rho_L(n, \epsilon_F)$ diverges exponentially. On the other hand, if α is a Liouville number, $\sigma(H) \cap B$ (if it is nonempty) is shown to be singular continuous for a.e. λ.[27] This fact contrasts with Theorem 5.1.

The proof of (1) in Theorem 5.1 follows from positivity of the Lyapunov exponent due to the Combes–Thomas argument[28] which is roughly given by the distance between ϵ_F and $\sigma(H)$. The proof of (3) in Theorem 5.1 follows from Remark 3.3.

We study some properties of $\rho_L(n, \epsilon_F)$ further assuming some properties on the behavior of tr $W(q_n, \epsilon_F)$. We first assume

$$\lim_{n \to \infty} a_n(\alpha) = \infty, \tag{5.3}$$

in which case the quasiperiodic potential $V(j)$ is "close to periodic" so that we expect the argument in Theorem 3.1 would be useful.

Proposition 5.1. Let $\theta = 0$, $\epsilon_F \in \sigma(H)$ and assume (5.3).

(1) If $\limsup_{n\to\infty} |\mathrm{tr}\, W(q_n, \epsilon_F)| > 2$, then $\limsup_{n\to\infty} \rho_L(n, \epsilon_F) = \infty$.

(2) If $\liminf_{n\to\infty} |\mathrm{tr}\, W(q_n, \epsilon_F)| < 2$, and $\{\rho_L(q_n, \epsilon_F)\}_{n=1}^{\infty}$ is bounded, then $\liminf_{n\to\infty} \rho_L(n, \epsilon_F) = 0$.

To prove Proposition 5.1, we prepare

Lemma 5.1. Let $\lambda, \lambda^{-1} \in \mathbb{C}$ $(|\lambda| \geqslant 1)$ be eigenvalues of $W(q_n, \epsilon_F)$.

(1) If $|\mathrm{tr}\, W(q_n, \epsilon_F)| < 2$, then

$$b(kq_n, \epsilon_F) = \frac{\sin k\theta}{\sin \theta}\, b(q_n, \epsilon_F), \qquad 1 \leqslant k \leqslant a_{n+1}(\alpha),$$

where $\theta \in \mathbb{R}$ is determined by $\lambda = e^{i\theta}$.

(2) If $|\mathrm{tr}\, W(q_n, \epsilon_F)| > 2$, then

$$b(kq_n, \epsilon_F) = \frac{\sinh k\theta}{\sinh \theta}\, b(q_n, \epsilon_F), \qquad 1 \leqslant k \leqslant a_{n+1}(\alpha),$$

where $\theta > 0$ is determined by $\lambda = (\mathrm{sgn}\, \lambda)\, e^{\theta}$.

Proof. Since $V(q_n + k) = V(k)$, $(1 \leqslant k \leqslant q_{n+1} - 2,\ n \geqslant 1)$,[14] we have $W(kq_n, \epsilon_F) = W(q_n, \epsilon_F)^k$, $k = 1, 2,..., a_{n+1}$. Hence the proof reduces to the computation of some powers of matrices which is done in the proof of Theorem 3.1. ▮

Proof of Proposition 5.1.

(1) By assumption, there exists $\delta > 0$ and a subsequence $n' = n(k)$ such that $|\mathrm{tr}\, W(q_n, \epsilon_F)| > 2 + \delta$. We rewrite n' as n. Let $\lambda(n)$ ($|\lambda(n)| > 1$) be an eigenvalue of $W(q_n, \epsilon_F)$. We consider the triangulation of $W(q_n, \epsilon_F)$ by an unitary matrix $U(n)$

$$V(n, \epsilon_F) := U^*(n)\, W(q_n, \epsilon_F)\, U(n) = \begin{pmatrix} \lambda(n) & c(n) \\ 0 & \lambda(n)^{-1} \end{pmatrix}, \qquad c(n) \in \mathbb{C}.$$

Then

$$V(n, \epsilon_F)^*\, V(n, \epsilon_F) = \begin{pmatrix} |\lambda(n)|^2 & \overline{\lambda(n)}\, c(n) \\ \overline{c(n)}\, \lambda(n) & |\lambda(n)|^{-2} + |c(n)|^2 \end{pmatrix}$$

and hence

$$\|W(q_n, \epsilon_F)\|_{\mathrm{HS}}^2 \geqslant |\lambda(n)|^2 + |\lambda(n)|^{-2} = (\mathrm{tr}\, W(q_n, \epsilon_F))^2 - 2.$$

By (2.4), $\rho_L(q_n, \epsilon_F) = \frac{1}{4}\,(\|W(q_n, \epsilon_F)\|_{\mathrm{HS}}^2 - 2) > \delta_1$ for some $\delta_1 > 0$. By Lemma 5.1(2),

$$\rho_L(kq_n, \epsilon_F) = \left(\frac{\sinh k\theta(n)}{\sinh \theta(n)}\right)^2 \rho_L(q_n, \epsilon_F) > \left(\frac{\sinh k\theta(n)}{\sinh \theta(n)}\right)^2 \delta_1,$$

$(1 \leqslant k \leqslant a_{n+1}(\alpha))$. $\theta(n) > 0$ is determined by $\lambda(n) = (\mathrm{sgn}\, \lambda(n))\, e^{\theta(n)}$. Since $|\lambda(n) + \lambda(n)^{-1}| = |\mathrm{tr}\, W(q_n, \epsilon_F)| > 2 + \delta$, $|\lambda(n)| - 1$ is bounded from below and thus $\theta(n) > \delta_2 > 0$ for some $\delta_2 > 0$. On the other hand, since $\epsilon_F \in \sigma(H)$,

$|\mathrm{tr}\, W(q_n, \epsilon_F)|$ is bounded (Remark 5.1(2)). Therefore $|\theta(n)|$ is bounded so that $|\sinh \theta(n)| < C_1$ for some constant $C_1 > 0$. Hence

$$\rho_L(a_{n+1}(\alpha)\, q_n, \epsilon_F) > \frac{\delta_1}{C_1^2} \sinh^2(a_{n+1}(\alpha)\, \delta_2).$$

Since $\lim_{n\to\infty} a_n(\alpha) = \infty$, the result follows.

(2) Let $\theta(n) \in (0, \pi)$ determined by $\lambda(n) = e^{i\theta(n)}$. There exists $\theta \in [0, \pi]$ and a subsequence $\theta(n')$ such that $\theta(n') \to \theta$ as $n \to \infty$. Due to the assumption $\liminf_{n\to\infty} |\mathrm{tr}\, W(q_n, \epsilon_F)| < 2$, we can assume $\theta \in [\delta_1, \pi - \delta_1]$ for some $\delta_1 > 0$. We rewrite $\theta(n)$ instead of $\theta(n')$. Let r_k/s_k, $r_k \in \mathbb{N}$, $s_k \in \mathbb{N}$ be the Diophantine approximation of θ/π. Then we have[20]

$$\frac{\theta}{\pi} \in \left(\frac{r_k - \frac{1}{s_{k+1}}}{s_k}, \frac{r_k + \frac{1}{s_{k+1}}}{s_k} \right).$$

We fix $k \in \mathbb{N}$ arbitrary. Since $\theta(n) \to \theta$, there exists $N = N(k) \in \mathbb{N}$ such that if $n \geqslant N$, we have

$$\frac{\theta(n)}{\pi} \in \left(\frac{r_k - \frac{1}{s_{k+1}}}{s_k}, \frac{r_k + \frac{1}{s_{k+1}}}{s_k} \right).$$

If $\theta/\pi \in \mathbb{Q}$, then $\theta/\pi = p/q$ for some $p, q \in \mathbb{N}$ and

$$\frac{\theta(n)}{\pi} \in \left(\frac{p - \epsilon'}{q}, \frac{p + \epsilon'}{q} \right),$$

where $\epsilon' > 0$ can be taken arbitrary small, when n is large enough. Then the rest of this proof also works. Now we have

$$\frac{s_k \theta(n)}{\pi} \in \left(r_k - \frac{1}{s_{k+1}}, r_k + \frac{1}{s_{k+1}} \right).$$

We fix $\epsilon > 0$ arbitrary small. Since $\lim_{k\to\infty} s_k = \infty$, by taking k sufficiently large, we have $\sin s_k \theta(n) \in (-\epsilon, \epsilon)$, $n \geqslant N(k)$. Since $\theta(n)$ converges to $\theta \in [\delta_1, \pi - \delta_1]$, $|\sin \theta(n)| > \delta_2 > 0$ for some $\delta_2 > 0$. By Lemma 5.1(1),

$$\rho_L(l q_n, \epsilon_F) = \left| \frac{\sin l\theta(n)}{\sin \theta(n)} \right|^2 |b(n, \epsilon_F)|^2,$$

for $1 \leqslant l \leqslant a_{n+1}(\alpha)$. By assumption, $|b(n, \epsilon_F)| < C_2$ for some $C_2 > 0$. Thus if we could let $l = s_k$, then

$$\rho_L(s_k q_n, \epsilon_F) \leqslant \frac{C_2^2}{\delta_2^2} |\sin s_k \theta(n)|^2 \leqslant \frac{C_2^2 \epsilon^2}{\delta_2^2}.$$

However, because $\lim_{n\to\infty} a_n(\alpha) = \infty$, $a_{n+1}(\alpha) \geqslant s_k$ for sufficiently large n. ∎

In what follows, we construct examples which satisfy the assumption of Proposition 5.1(2). Let $M(0, E) := T(0, E)$, $M(n, E) := W(q_n, E)$, $r(n, E) := \operatorname{tr} M(n, E)$. $M(n, E)$ is known to satisfy the following recursive equation.[14]

$$M(n+1, E) = M(n-1, E)\, M(n, E)^{a_{n+1}}, \qquad n \geqslant 1.$$

Since $M(j, E) \in \mathscr{G}$, we can write

$$M(n, E) = \begin{pmatrix} \alpha_n & \beta_n \\ \overline{\beta_n} & \overline{\alpha_n} \end{pmatrix}, \qquad \alpha_n, \beta_n \in \mathbb{C}, \quad |\alpha_n|^2 - |\beta_n|^2 = 1.$$

The following lemma is Lemma 4.1 in ref. 16 which we also present here for the sake of completeness.

Lemma 5.2. Fix $E \in \mathbb{R}$. Suppose $a_k \in \mathbb{N}$, $k = 1,\dots, n$, $C > 0$, $R \in \mathbb{N}$, $\delta > 0$ are given which satisfy

(i) $|\beta_k| \leqslant C \prod_{l=1}^{k} (1 + \frac{1}{2^l})$, $k = 1,\dots, n$.

(ii) $|r(k, E)| < 2 - 2\delta_k$, $R \leqslant k \leqslant n$, $\delta_k = \delta(\frac{1}{2} + \frac{1}{2^k})$.

Then we can choose $a_{n+1} \in \mathbb{N}$ such that $M(n+1, E) := M(n-1, E) \times M(n, E)^{a_{n+1}}$ satisfies

(1) $|\beta_{n+1}| \leqslant C \prod_{l=1}^{n+1} (1 + \frac{1}{2^l})$,

(2) $|r(n+1, E)| < 2 - 2\delta_{n+1}$.

Moreover, we have $\lim_{n\to\infty} a_n = \infty$.

Proof. Take $\theta_n \in (0, \pi)$ such that $2\cos\theta_n = r(n, E)$. Pick $a_{n+1} \in \mathbb{N}$ and let $M(n+1, E) := M(n-1, E)\, M(n, E)^{a_{n+1}}$. Then

$$M(n+1, E) = \frac{\sin a_{n+1}\theta_n}{\sin\theta_n} M(n-1, E)\, M(n, E) - \frac{\sin(a_{n+1}-1)\,\theta_n}{\sin\theta_n} M(n-1, E).$$

Set $C_k = C \prod_{l=1}^{k} (1+\frac{1}{2^l})$. It is easy to see

$$|\beta_{n+1}| \leqslant C_n\{(2\sqrt{1+C_n^2}+1)\,|\sin a_{n+1}\theta_n| + |\sin \theta_n|\}\,\frac{1}{|\sin \theta_n|},$$

$$|\Re\alpha_{n+1}| \leqslant \left|\frac{\sin a_{n+1}\theta_n}{\sin \theta_n}\right| (2+2C_n^2) + 1 - \delta_{n-1}.$$

Take $\epsilon_n', \epsilon_n'' > 0$ such that

$$(2\sqrt{1+C_n^2}+1)\,\epsilon_n' < |\sin \theta_n|\,\frac{1}{2^{n+1}},$$

$$(2+2C_n^2)\,\frac{\epsilon_n''}{|\sin \theta_n|} < \frac{\delta}{2^{n+1}}.$$

Set $\epsilon_n = \min\{\epsilon_n', \epsilon_n''\}$ and take $a_{n+1} \in \mathbb{N}$, $a_{n+1} \geqslant n+1$ such that $|\sin a_{n+1}\theta_n| < \epsilon_n$. Then β_{n+1} and $r(n+1, E)$ satisfy (1) and (2) in the statement of Lemma 5.2 respectively. ▮

Remark 5.2. An example which satisfies the hypothesis of Lemma 5.2 is: $\lambda_2 = 0$, $R = 0$, and $\epsilon_F \in \{k^2 \notin \mathscr{E} : |2\cos a_1 k + \frac{\lambda_1}{k}\sin a_1 k| < 2\}$.

Take $\epsilon_F > 0$ which satisfies the hypothesis of Lemma 5.2 and let $\alpha = [0, a_1, a_2, \ldots] \in \mathbb{Q}^c \cap (0, 1)$ be corresponding irrational number associated with $\{a_n\}_{n\in\mathbb{N}}$ given by Lemma 5.2. Then $|\mathrm{tr}\, W(q_n, \epsilon_F)| < 2 - \delta$, $\rho_L(q_n, \epsilon_F) \leqslant C^2 \prod_{n=1}^{\infty} (1+\frac{1}{2^n})^2 < \infty$, and thus α, ϵ_F satisfy the assumption of Proposition 5.1(2). Therefore $\liminf_{n\to\infty} \rho_L(n, \epsilon_F) = 0$ (in this case, $\rho_L(n, \epsilon_F)$ has at most polynomial growth even if α is not a bounded density number). This example tells us, if $\lim_{n\to\infty} a_n(\alpha) = \infty$, we can not expect to have lower bound of $\rho_L(n, \epsilon_F)$. We note, however, that $\sum_{n=1}^{l} \rho_L(n, \epsilon_F) \geqslant Cl^{\gamma}$ for some $\gamma > 0$ is shown[18, 19, 29] when $q_n \leqslant C^n$ for some $C > 0$, where $\gamma > 0$ depends on λ_1, λ_2, ϵ_F, and $C > 0$.

6. CONCLUDING REMARKS

We studied and reviewed the behavior of the Landauer resistivity as sample size tends to infinity when the potential is periodic, random, and quasiperiodic. The results are summarized as follows.

(1) When $\epsilon_F \in \rho(H)$, the Landauer resistivity diverges exponentially due to positivity of the Lyapunov exponent. This implies the conductivity vanishes in such cases, which is consistent with well-known result of the

band theory in solid state physics. On the other hand, if $\epsilon_F \in \sigma(H)$, situation becomes different.

(2) in the case of periodic potential where $\sigma(H)$ is absolutely continuous, the Landauer resistivity is bounded but does not converge if $\epsilon_F \in \sigma(H)$ unless it diverges like n^2. In this case $\overline{\{\rho_L(n, \epsilon_F) : n \in \mathbb{N}\}}$ is a closed interval or finitely many discrete points which is not stable under the small variation of ϵ_F, reflecting the behavior of corresponding Bloch waves.

(3) in the case of random potential where $\sigma(H) \setminus \mathscr{E}$ is pure point with exponentially decaying eigenfunctions, the Landauer resistivity diverges exponentially, which implies zero conductivity. This fact is consistent with calculations by Kubo-type formula done in Anderson model.[8–11]

(4) in the case of quasiperiodic potential where $\sigma(H)$ is generically purely singular continuous, it is known that the Landauer resistivity grows at most polynomial order. We found examples where $\liminf_{n\to\infty} \rho_L(n, \epsilon_F)$ is equal to zero.

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